Second kind odd radial Mathieu function

$$Yo_{2n+p}(h, \cosh u) = \frac{\sqrt{\frac{\pi}{2}}}{B_{2-p}} \sum_{m=0}^{\infty} (-1)^{n-m} B_{2m+p} \left[Y_{m+1}\left(\frac{1}{2} he^{u}\right) J_{m-1+p} \cdot \left(\frac{1}{2} he^{-u}\right) - Y_{m-1+p}\left(\frac{1}{2} he^{u}\right) J_{m+1}\left(\frac{1}{2} he^{-u}\right)\right] \tag{31}$$

Where $p \in \{0, 1\}$ and J_m, Y_m are the conventional Bessel functions.

The normalization constants are

$$M^e_{2n}(h) = \int_0^{2\pi} [Se_{2n}(h, \cos v)]^2 \, dv = 2\pi \sum_{m=0}^{\infty} A^2_{2m}/\sigma_m: \qquad \sigma_m = \begin{cases} 1 & m = 0 \\ 2 & m \neq 0 \end{cases}$$

$$M^e_{2n+1}(h) = \int_0^{2\pi} [Se_{2n+1}(h, \cos v)]^2 \, dv = \pi \sum_{m=0}^{\infty} A^2_{2m+1}$$

$$M^o_{2n+p}(h) = \int_0^{2\pi} [So_{2n+p}(h, \cos v)]^2 \, dv = \pi \sum_{m=1}^{\infty} B^2_{2m+p}.$$

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On The Modal Expansion of Resonator Field in the Source Region

A. S. Omar, E. Jensen, and S. Lütgert

***Abstract*—Two field expansions for the electromagnetic field radiated by electric and magnetic currents in a cavity resonator are presented. The first utilizes the cavity resonant modes only, while the other utilizes, in addition, the irrotational modes. The first expansion is shown to be more suitable if the exciting currents have volume distributions. On the other hand, the second expansion is more suitable if the resonator contains surface or filamentary current distributions. Typical examples are given to demonstrate the convergence behavior of the two expansions near and within the source region.**

Introduction

In a field-theoretical analysis of microwave tubes, e.g., klystrons, magnetrons, travelling wave tubes, gyrotrons, and orotrons, one can divide the describing equations of the structure into two systems of equations. The first system expresses the electromagnetic field in terms of the exciting current(s). It is just Maxwell's equations with source terms. This system is linear if a small signal approximation is considered or if the nonlinear materials are replaced by polarization currents which can be added to the excitation ones. The second system describes the influence of the electromagnetic field on the motion of the electrons. It expresses then the exciting current(s) in terms of the excited field. This system is usually nonlinear except for the small signal analysis. The two systems must be solved simultaneously. They can be considered to represent a feedback system with a linear forward transmission and a nonlinear backward transmission. Well-established methods of control theory can consequently be applied to this feedback system in order to study the featuring characteristics like stability, starting and sustaining oscillation conditions, modulation, noise performance, etc.

The analysis of the linear system can be done in either time or frequency domain. In this paper, the analysis will be conducted in frequency domain. If time-domain information are needed, e.g., for the nonlinear system, an inverse Fourier transform must be made. Because the interaction between the electron beam and the electromagnetic wave in most of the microwave tubes takes place inside a cavity resonator, which may be either partially or completely shielded, the excited electromagnetic field can be expressed as expansions in terms of the empty cavity modes. These modes can be classified into divergence-free modes (which are the cavity resonant modes) and curl-free (or irrotational) modes.

The accuracy and convergence of these expansions is particularly important within the electron beam, i.e., in the source region, because accurate expressions for the electromagnetic field within the beam are necessary for the accurate solution of the nonlinear system (i.e., the electrons' equations of motion). It is the aim of this letter to study the different possible expansions along with their accuracy and convergence in the source region.

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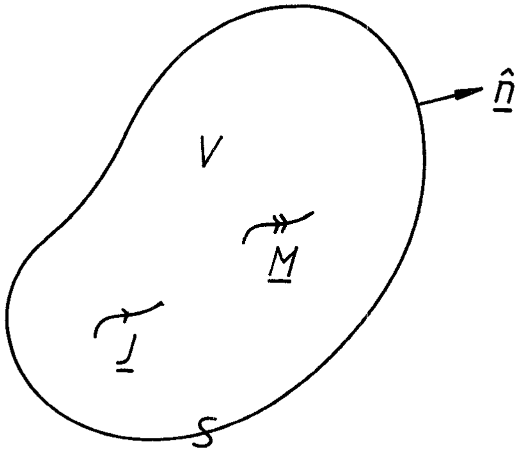

Fig. 1. A cavity resonator excited by an electric current $\boldsymbol{J}$ and a magnetic current $\boldsymbol{M}$.

Basic Formulation

Consider the cavity resonator shown in Fig. 1. The closed surface S is assumed to be perfectly conducting. Coupling apertures have been short-circuited are replaced by equivalent magnetic surface currents. The cavity is excited by an electric current $\boldsymbol{J}$ and a magnetic current $\boldsymbol{M}$. These include convection currents (electron beam), polarization currents, which replace dielectric or magnetic inserts, as well as equivalent surface currents, which replace coupling apertures or metal inserts.

The cavity can now be considered empty. It has a tribly-infinite set of resonant modes $\{\boldsymbol{E}_n, \boldsymbol{H}_n\}$, which are characterized by

$$\nabla \times \boldsymbol{E}_n = -j\omega_n \mu_o \boldsymbol{H}_n \quad (1a)$$

$$\nabla \times \boldsymbol{H}_n = j\omega_n \epsilon_o \boldsymbol{E}_n \quad (1b)$$

$$\epsilon_o \int_V \boldsymbol{E}_n \cdot \boldsymbol{E}_m^* \, dV = \mu_o \int_V \boldsymbol{H}_n \cdot \boldsymbol{H}_m^* \, dV = W_n \delta_{nm} \quad (1c)$$

where ω_n is the resonant frequency of the nth resonant mode, and δ_{nm} is the Kronecker delta.

The set of resonant modes, which are divergence-free, is not a complete one. For the expansion of an arbitrary electromagnetic field inside the cavity, one has to complete the resonant modes with the irrotational modes $\{\boldsymbol{F}_n, \boldsymbol{G}_n\}$ [1], which are characterized by

$$\boldsymbol{F}_n = \nabla\varphi_n, \quad \hat{\boldsymbol{n}} \times \boldsymbol{F}_n|_S = 0 \quad (2a)$$

$$\epsilon_o \int_V \boldsymbol{F}_n \cdot \boldsymbol{F}_m^* \, dV = U_n \, \delta_{nm} \quad (2b)$$

$$\boldsymbol{G}_n = \nabla\psi_n, \quad \hat{\boldsymbol{n}} \cdot \boldsymbol{G}_n|_S = 0 \quad (3a)$$

$$\mu_o \int_V \boldsymbol{G}_n \cdot \boldsymbol{G}_m^* \, dV = V_n \, \delta_{nm} \quad (3b)$$

The electromagnetic field inside the resonator satisfies Maxwell's equations

$$\nabla \times \boldsymbol{E} = -(\boldsymbol{M} + j\omega\mu_o \boldsymbol{H}) \quad (4a)$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + j\omega\epsilon_o \boldsymbol{E} \quad (4b)$$

A. Field Expansion in Terms of $\{\boldsymbol{E}_n\}$ and $\{\boldsymbol{H}_n\}$

The two quantities $\boldsymbol{J} + j\omega\epsilon_o \boldsymbol{E}$ and $\boldsymbol{M} + j\omega\mu_o \boldsymbol{H}$ are divergence-free. They can then be expanded with respect to $\{\boldsymbol{E}_n\}$ and $\{\boldsymbol{H}_n\}$, respectively, by

$$\boldsymbol{J} + j\omega\epsilon_o \boldsymbol{E} = j\omega\epsilon_o \sum_n \frac{a_n}{\sqrt{W_n}} \boldsymbol{E}_n \quad (5a)$$

$$\boldsymbol{M} + j\omega\mu_o \boldsymbol{H} = j\omega\mu_o \sum_n \frac{b_n}{\sqrt{W_n}} \boldsymbol{H}_n \quad (5b)$$

Substituting (5) in (4) and making use of (1) results in

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \frac{\omega_n}{\omega^2 - \omega_n^2} \begin{bmatrix} \omega_n & \omega \\ \omega & \omega_n \end{bmatrix} \begin{bmatrix} C_n^{(e)} \\ C_n^{(h)} \end{bmatrix} \quad (6)$$

where

$$C_n^{(e)} = \frac{-1}{j\omega\sqrt{W_n}} \int_V \boldsymbol{J} \cdot \boldsymbol{E}_n^* \, dV \quad (7a)$$

$$C_n^{(h)} = \frac{-1}{j\omega\sqrt{W_n}} \int_V \boldsymbol{M} \cdot \boldsymbol{H}_n^* \, dV \quad (7b)$$

B. Field Expansion in Terms of $\{\boldsymbol{E}_n, \boldsymbol{F}_n\}$ and $\{\boldsymbol{H}_n, \boldsymbol{G}_n\}$

The electric and magnetic fields are now expanded as

$$\boldsymbol{E} = \sum_n \frac{e_n}{\sqrt{W_n}} \boldsymbol{E}_n + \sum_n \frac{f_n}{\sqrt{U_n}} \boldsymbol{F}_n \quad (8a)$$

$$\boldsymbol{H} = \sum_n \frac{h_n}{\sqrt{W_n}} \boldsymbol{H}_n + \sum_n \frac{g_n}{\sqrt{V_n}} \boldsymbol{G}_n \quad (8b)$$

Substituting the above expansions in (4) and making use of (1), (2), and (3) results in

$$\begin{bmatrix} e_n \\ h_n \end{bmatrix} = \frac{\omega}{\omega^2 - \omega_n^2} \begin{bmatrix} \omega & \omega_n \\ \omega_n & \omega \end{bmatrix} \begin{bmatrix} C_n^{(e)} \\ C_n^{(h)} \end{bmatrix} \quad (9)$$

$$f_n = \frac{-1}{j\omega\sqrt{U_n}} \int_V \boldsymbol{J} \cdot \boldsymbol{F}_n^* \, dV \quad (10a)$$

$$g_n = \frac{-1}{j\omega\sqrt{V_n}} \int_V \boldsymbol{M} \cdot \boldsymbol{G}_n^* \, dV \quad (10b)$$

C. Excitation by an Electric Current Only ($\boldsymbol{M} = 0$)

In order to simplify the analysis, we assume that the cavity is excited by an electric current $\boldsymbol{J}$ only. This results in the following expansions for the electric and magnetic fields:

$$\boldsymbol{E} = \sum_n \frac{\omega_n^2}{\omega^2 - \omega_n^2} \frac{C_n^{(e)}}{\sqrt{W_n}} \boldsymbol{E}_n - \frac{\boldsymbol{J}}{j\omega\epsilon_o} \quad (11a)$$

$$= \sum_n \frac{\omega^2}{\omega^2 - \omega_n^2} \frac{C_n^{(e)}}{\sqrt{W_n}} \boldsymbol{E}_n + \sum_n \frac{f_n}{\sqrt{U_n}} \boldsymbol{F}_n \quad (11b)$$

$$\boldsymbol{H} = \sum_n \frac{\omega\,\omega_n}{\omega^2 - \omega_n^2} \frac{C_n^{(e)}}{\sqrt{W_n}} \boldsymbol{H}_n \quad (12)$$

Comparing the two $\boldsymbol{E}_n$-series in (11a) and (11b), one can easily show that the convergence in (11a) is worse than that in (11b) (note that $\omega_n^2 \sim n^2$ as $n \to \infty$). The advantage of (11a), on the other hand, is the absence of the set $\{\boldsymbol{F}_n\}$ which must additionally be determined.

Let us investigate now the case with the exciting current $\boldsymbol{J}$ being a surface distribution. The term $(\boldsymbol{J}/j\omega\epsilon_o)$ in (11a) has then a dirac-delta dependence in the direction normal to the surface current. The electric field, on the other hand, must be continuous across the current sheet. This means that a part of the $\boldsymbol{E}_n$-series in (11a) must compensate the dirac-delta dependence of the term $(\boldsymbol{J}/j\omega\epsilon_o)$, while the rest of the series converges to the actual (continuous) electric field. The $\boldsymbol{E}_n$-series in (11a) near and across the current sheet does consequently not converge. A very strong Gibb's effect accompanies the series there [2]. The situation is much worse if the current source is a filamentary one.

If the convergence of the series in (12) is investigated, it is easily shown that the series has a step discontinuity at the current sheet (the discontinuity in the tangential magnetic field is equal to the surface electric current). The $\boldsymbol{E}_n$-series in (11b), which converges better than that in (12) (note that $\omega_n \sim n$ as $n \to \infty$), converges uniformly across the current sheet. The $\boldsymbol{F}_n$-series in (11b) also converges uniformly across the current sheet because the sum of the two series (the electric field) is continous there.

We can conclude that the two series in (11b) converge uniformly within the source region, even if the source were a current sheet or a current filament. On the other hand, the series in (11a) converges only if the current source is a well-behaved "volume" distribution. For the latter case, the expansion of the electric field in (11a) is advantageous because it is free from the set $\{\boldsymbol{F}_n\}$, although its convergence is still slower than that of the two series in (11b).

Numerical Results

In order to compare the convergence of the two series in (11a) and (11b), the excitation of a rectangular cavity resonator has been investigated. The cartesian coordinates system is chosen such that the points (0.0, 0.0, 0.0), (1.0, 0.0, 0.0), (1.0, 0.5, 0.0), (0.0, 0.5, 0.0), (0.0, 0.0, 1.2), (1.0, 0.0, 1.2), (1.0, 0.5, 1.2), and (0.0, 0.5, 1.2) define the vertices of the cavity. The y-directed exciting current is rotationally symmetric with respect to the line connecting the two points (0.5, 0.0, 0.6) and (0.5, 0.5, 0.6). It is given by

$$J_y = \frac{1}{\pi\sigma^2} \cos(2\pi y) \exp\left(-\frac{(x-0.5)^2 + (z-0.6)^2}{\sigma^2}\right) \quad (13)$$

Two cases have been investigated. The first with $\sigma = 10^{-4}$, which resembles a y-directed filamentary current in the middle of the cavity. Fig. 2(a) shows the x-dependence of E_y at $y = 0.15$ and $z = 0.6$ once according to (11a), which is the solid line, and once according to (11b), which is the dashed line. The operating frequency has been chosen to be much smaller than the resonance frequency of the lowest resonant mode (TE^z_{101}). The expansion according to (11a) shows a strong Gibb's effect and hence does not converge at all. On the other hand, the expansion according to (11b) is a well-behaved function. Fig. 2(b) is similar to Fig. 2(a) but with the operating frequency very near to the resonance frequency of the TE^z_{111}-resonant mode. The field expansion according to (11b) (the dashed line) converges to the field distribution of the TE^z_{111}-resonant mode (which is a half sine-wave). The expansion according to (11a) (the solid line) converges to the same distribution only far from the location of the filamentary source. Near and at the source, the expansion according to (11a) shows a strong Gibb's effect and does not converge.

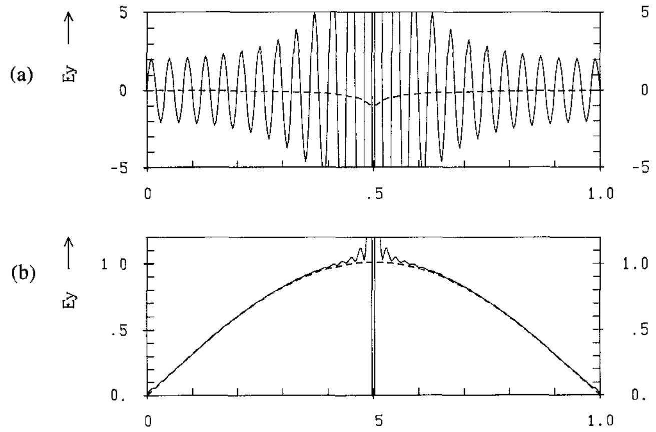

Fig. 2. The x-dependence of E_y excited by a filamentary current J_y (a) Far from resonance. (b) Very near to resonance. ——according to (11a). --------according to (11b).

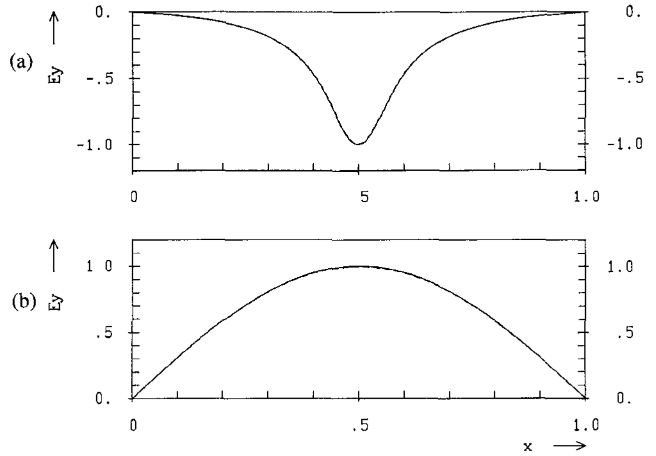

Fig. 3. The x-dependence of E_y excited by a y-directed current J_y with a Gaussian distribution in the x-z-plane. (a) Far from resonance. (b) Very near to resonance. ——according to (11a). --------according to (11b).

The second case corresponds to $\sigma = 0.05$, which resembles a Gaussian volume distribution. Fig. 3(a) and Fig. 3(b) show the field expansions corresponding to an operating frequency which is far from and very near to the resonance frequency of the TE^z_{111}-resonant mode, respectively. For this case, the two expansions according to (11a) (the solid line) and according to (11b) (the dashed line) are indistinguishable, which means that the convergence of the two series is quite similar.

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